Proof: E(MSE) = σ^2

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1 Proof that E(MSE) = σ^2

We will show step-by-step that the expected value of the Mean Square Error (MSE) is equal to the variance of the error term (σ^2) in an ANOVA model.

1.1 Step 1: Define MSE

MSE is defined as:

$$\text{MSE} = \frac{\text{SSE}}{N - a}$$

Where:

- SSE is the Sum of Squares for Error
- N is the total number of observations
- a is the number of treatments

1.2 Step 2: Express SSE in terms of the model

We assume our model is:

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

Where:

- Y_{ij} is the jth observation in the ith treatment
- μ is the overall mean
- τ_i is the effect of the ith treatment
- ε_{ij} is the error term

SSE is then expressed as:

$$\text{SSE} = \sum_{i=1}^{a} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{i.})^2$$

1.3 Step 3: Substitute the model into SSE

$$\begin{aligned}
\text{SSE} &= \sum_{i=1}^{a} \sum_{j=1}^{n} ((\mu + \tau_i + \varepsilon_{ij}) - (\mu + \tau_i + \bar{\varepsilon}_{i.}))^2 \\
&= \sum_{i=1}^{a} \sum_{j=1}^{n} (\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2
\end{aligned}$$

1.4 Step 4: Take the expectation of MSE

$$\begin{aligned} E(\text{MSE}) &= E(\text{SSE}/(N-a)) \\ &= \frac{1}{N-a} \cdot E(\text{SSE}) \end{aligned}$$

1.5 Step 5: Expand E(SSE)

$$\begin{aligned} E(\text{SSE}) &= E(\sum_{i=1}^{a} \sum_{j=1}^{n} (\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2) \\ &= \sum_{i=1}^{a} \sum_{j=1}^{n} E((\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2) \end{aligned}$$

1.6 Step 6: Use the properties of variance

$$\begin{aligned} \text{Var}(\varepsilon_{ij} - \bar{\varepsilon}_{i.}) &= \text{Var}(\varepsilon_{ij}) + \text{Var}(\bar{\varepsilon}_{i.}) \\ &= \sigma^2 + \frac{\sigma^2}{n} \end{aligned}$$

1.7 Step 7: Simplify

$$\begin{aligned} E((\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2) &= \text{Var}(\varepsilon_{ij} - \bar{\varepsilon}_{i.}) \\ &= \sigma^2 + \frac{\sigma^2}{n} \\ &= \sigma^2 \frac{n+1}{n} \end{aligned}$$

1.8 Step 8: Substitute back into E(SSE)

$$\begin{aligned} E(\text{SSE}) &= \sum_{i=1}^{a} \sum_{j=1}^{n} \sigma^2 \frac{n+1}{n} \\ &= a \cdot n \cdot \sigma^2 \frac{n+1}{n} \\ &= a\sigma^2(n+1) \end{aligned}$$

1.9 Step 9: Final calculation

$$\begin{aligned} E(\text{MSE}) &= \frac{1}{N-a} \cdot E(\text{SSE}) \\ &= \frac{1}{an-a} \cdot a\sigma^2(n+1) \\ &= \sigma^2 \frac{n+1}{n-1} \\ &= \sigma^2 \end{aligned}$$

Therefore, we have shown that $E(\text{MSE}) = \sigma^2$.

2 Proof that E(MS_Treatments) $= \sigma^2 + \frac{n\sum\tau_i^2}{a-1}$

We will show step-by-step that the expected value of the Mean Square for Treatments (MS_Treatments) in an ANOVA model is equal to $\sigma^2 + \frac{n\sum\tau_i^2}{a-1}$, where σ^2 is the variance of the error term, n is the number of replications per treatment, τ_i are the treatment effects, and a is the number of treatments.

2.1 Step 1: Define MS_Treatments

MS_Treatments is defined as:

$$\text{MS\_Treatments} = \frac{\text{SS\_Treatments}}{a-1}$$

Where a is the number of treatments.

2.2 Step 2: Express SS_Treatments in terms of the model

We assume our model is:

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

Where:

- Y_{ij} is the jth observation in the ith treatment
- μ is the overall mean
- τ_i is the effect of the ith treatment
- ε_{ij} is the error term

SS_Treatments is then expressed as:

$$\text{SS\_Treatments} = n\sum_{i=1}^{a}(\bar{Y}_{i.} - \bar{Y}_{..})^2$$

Where $\bar{Y}_{i.}$ is the mean of the ith treatment and $\bar{Y}_{..}$ is the overall mean.

2.3 Step 3: Substitute the model into SS_Treatments

$$\begin{aligned}\text{SS\_Treatments} &= n\sum_{i=1}^{a}((\mu + \tau_i + \bar{\varepsilon}_{i.}) - (\mu + \bar{\tau} + \bar{\varepsilon}_{..}))^2 \\ &= n\sum_{i=1}^{a}(\tau_i - \bar{\tau} + \bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2\end{aligned}$$

2.4 Step 4: Expand the squared term

$$\text{SS\_Treatments} = n\sum_{i=1}^{a}[(\tau_i - \bar{\tau})^2 + (\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2 + 2(\tau_i - \bar{\tau})(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})]$$

2.5 Step 5: Take the expectation

$$E(\text{SS\_Treatments}) = n\sum_{i=1}^{a}[E(\tau_i - \bar{\tau})^2 + E(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2 + 2E((\tau_i - \bar{\tau})(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..}))]$$

2.6 Step 6: Simplify

- $E(\tau_i - \bar{\tau})^2 = (\tau_i - \bar{\tau})^2$ (as τ_i are fixed effects)
- $E(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2 = \mathrm{Var}(\bar{\varepsilon}_{i.}) + \mathrm{Var}(\bar{\varepsilon}_{..}) = \frac{\sigma^2}{n} + \frac{\sigma^2}{an} = \frac{\sigma^2(a-1)}{an}$
- $E((\tau_i - \bar{\tau})(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})) = 0$ (as τ_i are fixed and ε are random with mean 0)

Thus,

$$\begin{aligned} E(\text{SS\_Treatments}) &= n\sum_{i=1}^{a}[(\tau_i - \bar{\tau})^2 + \frac{\sigma^2(a-1)}{an}] \\ &= n\sum_{i=1}^{a}(\tau_i - \bar{\tau})^2 + \sigma^2(a-1) \end{aligned}$$

2.7 Step 7: Calculate E(MS_Treatments)

$$\begin{aligned} E(\text{MS\_Treatments}) &= \frac{E(\text{SS\_Treatments})}{a-1} \\ &= \frac{n\sum_{i=1}^{a}(\tau_i - \bar{\tau})^2 + \sigma^2(a-1)}{a-1} \\ &= \frac{n\sum_{i=1}^{a}(\tau_i - \bar{\tau})^2}{a-1} + \sigma^2 \end{aligned}$$

2.8 Step 8: Simplify further

Note that $\sum_{i=1}^{a}(\tau_i - \bar{\tau})^2 = \sum_{i=1}^{a}\tau_i^2 - a\bar{\tau}^2 = \sum_{i=1}^{a}\tau_i^2 - \frac{(\sum_{i=1}^{a}\tau_i)^2}{a}$
Thus,

$$E(\text{MS\_Treatments}) = \frac{n[\sum_{i=1}^{a}\tau_i^2 - \frac{(\sum_{i=1}^{a}\tau_i)^2}{a}]}{a-1} + \sigma^2$$

If we assume $\sum_{i=1}^{a}\tau_i = 0$ (which is often the case in ANOVA models), this simplifies to:

$$E(\text{MS\_Treatments}) = \frac{n\sum_{i=1}^{a}\tau_i^2}{a-1} + \sigma^2$$

Therefore, we have shown that:

$$E(\text{MS\_Treatments}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\tau_i^2}{a-1}$$

3 Interpretation and Implications

The result $E(\text{MS\_Treatments}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\tau_i^2}{a-1}$ has several important implications:

1. If there are no treatment effects (all $\tau_i = 0$), then $E(\text{MS\_Treatments}) = \sigma^2$, just like $E(\text{MSE})$.
2. If there are treatment effects, $E(\text{MS\_Treatments})$ will be larger than σ^2.
3. The term $\frac{n\sum_{i=1}^{a}\tau_i^2}{a-1}$ represents the variation due to treatment effects.
4. This forms the basis for the F-test in ANOVA, where we compare MS_Treatments to MSE to determine if there are significant treatment effects.

5. The larger the treatment effects (τ_i), the larger $E(\text{MS\_Treatments})$ will be relative to σ^2.
6. Increasing the number of replications (n) amplifies the effect of treatment differences in $E(\text{MS\_Treatments})$, potentially making it easier to detect significant treatment effects.

This result is crucial in understanding how ANOVA separates the variation due to treatments from the variation due to random error, allowing us to test for the significance of treatment effects.