Design and Analysis of Experiments

Lecture 6 - Latin squares, and Related Designs

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Outline

Randomized Complete Block Design

- In Lecture 5, we introduced the **randomized complete block design**.
- Purpose: To reduce residual error in an experiment.
- Method: By removing variability due to a known and controllable nuisance variable.
- **Blocking Principle**: Used to control for factors that could add noise to the results.

Other Designs Utilizing Blocking Principle

- There are several designs that utilize the blocking principle beyond the randomized complete block design.
- **Example**: An experiment studying the effects of five formulations of rocket propellant on burning rate.
- Each formulation is mixed from a batch of raw material only large enough for five formulations to be tested.
- Formulations are prepared by different operators, each with varying skills and experience.

Two Nuisance Factors in Design

- In this experiment, two nuisance factors need to be controlled:
  1. **Batches of Raw Material**: Small batch sizes restrict testing to five formulations per batch.
  2. **Operators**: Skill and experience of operators may vary significantly.
- Goal: To average out these nuisance factors.

Latin Square Design

- The appropriate design to address this problem is called the **Latin square design**.
- This design ensures each formulation is:
  1. Tested exactly once in each batch of raw material.
  2. Prepared exactly once by each of five operators.
- The resulting arrangement forms a square, shown in Table 4.9.

Latin Square Structure

- The design is a square arrangement where the five formulations (or treatments) are denoted by Latin letters A, B, C, D, and E.
- This structure explains the name: **Latin square**.

Latin Square Structure Design

TABLE 4.9

Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators				
	1	2	3	4	5
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

Latin Square Structure Design

- We see that both batches of raw material (rows) and operators (columns) are orthogonal to treatments.
- The Latin square design is used to eliminate two nuisance sources of variability;
- that is, it systematically allows blocking in two directions. Thus, the rows and columns actually represent two restrictions on randomization.
- In general, a Latin square for p factors, or a pxp Latin square, containing p rows and p columns.
- Each of the resulting p2 cells contains one of the p letters that corresponds to the treatments, and each letter occurs once and only once in each row and column. Some examples of Latin squares are

Latin Square Structure Design

Some examples of Latin squares are

4 × 4	5 × 5	6 × 6
A B D C	*A D B E C*	*A D C E B F*
B C A D	*D A C B E*	*B A E C F D*
C D B A	*C B E D A*	*C E D F A B*
D A C B	*B E A C D*	*D C F B E A*
	E C D A B	*F B A D C E*
		E F B A D C

Latin Square Model

The statistical model for a Latin square design is given by:

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

where:

- y_{ijk}: observation in the i-th row, j-th treatment, and k-th column,
- μ: overall mean,
- α_i: effect of the i-th row,
- τ_j: effect of the j-th treatment,
- β_k: effect of the k-th column,
- ϵ_{ijk}: random error.

Additivity of the Model

Note: This is an effects model with complete additivity — there is no interaction between rows, columns, and treatments.

- Each treatment appears exactly once in each row and column.
- Consequently, only two of the subscripts i, j, or k are needed to denote a particular observation.
- **Example (Rocket Propellant)**:
  - If $i = 2$ and $k = 3$, then $j = 4$ (formulation D).
  - If $i = 1$ and $j = 3$ (formulation C), then $k = 3$.

Analysis of Variance (ANOVA) in Latin Square Design

- The total sum of squares of the $N = p^2$ observations is partitioned into components for rows, columns, treatments, and error:

$$SS_T = SS_{\text{Rows}} + SS_{\text{Columns}} + SS_{\text{Treatments}} + SS_E$$

- Each component has its respective degrees of freedom:
  - Rows: $p - 1$
  - Columns: $p - 1$
  - Treatments: $p - 1$
  - Error: $(p - 2)(p - 1)$

Hypothesis Testing

- Under the assumption that ϵ_{ijk} is normally and independently distributed with mean 0 and variance σ^2, each sum of squares (SS) is a chi-square random variable upon division by σ^2.
- To test for differences in treatment means, we use:

$$F_0 = \frac{\text{MS}_{\text{Treatments}}}{\text{MS}_E}$$

  which is distributed as $F_{p-1,(p-2)(p-1)}$ under the null hypothesis.
- We can also test for no row or column effects using:

$$\frac{\text{MS}_{\text{Rows}}}{\text{MS}_E} \quad \text{or} \quad \frac{\text{MS}_{\text{Columns}}}{\text{MS}_E}$$

- **Note**: Rows and columns represent restrictions on randomization, so these tests may not be entirely appropriate.

ANOVA Computation

- The computational procedure for ANOVA in a Latin square design uses treatment, row, and column totals.
- The sums of squares can be calculated as an extension of the randomized complete block design (RCBD), with SS_{Rows} obtained from the row totals.

ANOVA Table Structure

Source	Degrees of Freedom	Sum of Squares	Mean Square
Treatments	$p-1$	SS_{Rows}	MS_{Rows}
Rows	$p-1$	SS_{Columns}	$\text{MS}_{\text{Columns}}$
Columns	$p-1$	$SS_{\text{Treatments}}$	$\text{MS}_{\text{Treatments}}$
Error	$(p-2)(p-1)$	SS_E	MS_E

ANOVA Computation

TABLE 4.10

Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$	p^2-1		

A standard Latin square

- A Latin square in which the first row and column consists of the letters written in alphabetical order is called a standard Latin square,
- A standard Latin square can always be obtained by writing the first row in alphabetical order and then writing each successive row as the row of letters just above shifted one place to the left.
- Table 4.13 summarizes several important facts about Latin squares and standard Latin squares.
- As with any experimental design, the observations in the Latin square should be taken in random order.
- As we see in Table 4.13, there are a large number of Latin squares of a particular size, so it is impossible to enumerate all the squares and select one randomly. The usual procedure is

TABLE 4.13

Standard Latin Squares and Number of Latin Squares of Various Sizes[a]

Size	3×3	4×4	5×5	6×6	7×7	$p \times p$
Examples of standard squares	*A B C* *B C A* *C A B*	*A B C D* *B C D A* *C D A B* *D A B C*	*A B C D E* *B A E C D* *C D A E B* *D E B A C* *E C D B A*	*A B C D E F* *B C F A D E* *C F B E A D* *D E A B F C* *E A D F C B* *F D E C B A*	*A B C D E F G* *B C D E F G A* *C D E F G A B* *D E F G A B C* *E F G A B C D* *F G A B C D E* *G A B C D E F*	*ABC . . . P* *BCD . . . A* *CDE . . . B* ⋮ *PAB . . .* (*P* − 1)
Number of standard squares	1	4	56	9408	16,942,080	—
Total number of Latin squares	12	576	161,280	818,851,200	61,479,419,904,000	$p!(p-1)! \times$ (number of standard squares)

[a]Some of the information in this table is found in Fisher and Yates (1953). Little is known about the properties of Latin squares larger than 7×7.

Latin Squares Design

- Latin squares can be useful in situations where the rows and columns represent factors the experimenter wishes to study.
- There are no randomization restrictions in Latin square designs.
- Three factors (rows, columns, and letters), each at p levels, can be investigated in only p^2 runs.
- This design assumes no interaction between the factors.

Replication of Latin Squares

- A disadvantage of small Latin squares is that they provide a relatively small number of error degrees of freedom.
- For example:
  - A 3×3 Latin square has only two error degrees of freedom.
  - A 4×4 Latin square has only six error degrees of freedom, and so forth.
- To increase the error degrees of freedom, it is often desirable to replicate Latin squares.

Methods of Replicating Latin Squares

- A Latin square may be replicated in several ways:
  1. Use the same batches and operators in each replicate.
  2. Use the same batches but different operators in each replicate (or, equivalently, use the same operators but different batches).
  3. Use different batches and different operators.
- The method of replication affects the analysis of variance (ANOVA).

ANOVA for Replicated Latin Squares, Case 1

- Consider case 1, where the same levels of the row and column blocking factors are used in each replicate.
- Let y_{ijkl} be the observation in row i, treatment j, column k, and replicate l.
- The total number of observations is $N = np^2$, where n is the number of replicates and p is the number of levels.

ANOVA for a Replicated Latin Square, Case 1

■ **TABLE 4.14**

Analysis of Variance for a Replicated Latin Square, Case 1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{np}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p+1)-3]$	$\frac{SS_E}{(p-1)[n(p+1)-3]}$	
Total	$\sum\sum\sum\sum y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

ANOVA for a Replicated Latin Square, Case 2

- Now consider case 2 and assume that new batches of raw material but the same operators are used in each replicate. Thus, there are now five new rows (in general, p new rows) within each replicate.
- The ANOVA is summarized in Table 4.15. Note that the source of variation for the rows really measures the variation between rows within the n replicates

ANOVA for a Replicated Latin Square, Case 2

TABLE 4.15

Analysis of Variance for a Replicated Latin Square, Case 2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)(np-1)$	$\frac{SS_E}{(p-1)(np-1)}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

ANOVA for a Replicated Latin Square, Case 3

- Finally, consider case 3, where new batches of raw material and new operators are used in each replicate.
- Now the variation that results from both the rows and columns measures the variation resulting from these factors within the replicates.
- The ANOVA is summarized in Table 4.16.

ANOVA for a Replicated Latin Square, Case 3

TABLE 4.16
Analysis of Variance for a Replicated Latin Square, Case 3

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{p}\sum_{l=1}^{n}\sum_{k=1}^{p} y_{..kl}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Columns}}}{n(p-1)}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p-1)-1]$	$\frac{SS_E}{(p-1)[n(p-1)-1]}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

The Graeco-Latin square design

- Consider a pxp Latin square, and superimpose on it a second pxp Latin square in which the treatments are denoted by Greek letters.
- If the two squares when superimposed have the property that each Greek letter appears once and only once with each Latin letter, the two Latin squares are said to be orthogonal, and the design obtained is called a Graeco-Latin square.
- An example of a 4x4 Graeco-Latin square is shown in Table 4.18.

The Graeco-Latin square design

TABLE 4.18
4 × 4 Graeco-Latin Square Design

	Column			
Row	**1**	**2**	**3**	**4**
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

Graeco-Latin Square Design

- The Graeco-Latin square design can be used to systematically control three sources of extraneous variability, i.e., to block in three directions.
- The design allows investigation of four factors:
  - Rows
  - Columns
  - Latin letters
  - Greek letters
- Each factor is at p levels in only p^2 runs.
- Graeco-Latin squares exist for all $p \geq 3$ except when $p = 6$.

Statistical Model for Graeco-Latin Square Design

- The statistical model for the Graeco-Latin square design is:

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + \delta_l + \epsilon_{ijkl} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, p \end{cases} \quad (1)$$

- where:
  - y_{ijkl}: Observation in row i and column l for Latin letter j and Greek letter k
  - α_i: Effect of the ith row
  - β_j: Effect of Latin letter treatment j
  - γ_k: Effect of Greek letter treatment k
  - δ_l: Effect of column l
  - ϵ_{ijkl}: Random error component, $NID(0, \sigma^2)$

Model for Graeco-Latin Square Design

- Only two of the four subscripts are needed to identify an observation completely.
- The analysis of variance (ANOVA) for this design is similar to that of a Latin square.

Orthogonality and Sum of Squares in Graeco-Latin Square

- Greek letters appear exactly once in each row and column and exactly once with each Latin letter.
- The Greek letter factor is orthogonal to rows, columns, and Latin letter treatments.
- Sum of squares due to the Greek letter factor can be computed from the Greek letter totals, further reducing experimental error.
- Computational details are illustrated in Table 4.19.

Hypothesis Testing and Rejection Region

- The null hypotheses of equal row, column, Latin letter, and Greek letter treatments are tested by dividing the corresponding mean square by the mean square error.
- The rejection region is the upper tail point of the $F_{p-1,(p-3)(p-1)}$ distribution.

ANOVA for a Graeco-Latin Square Design

TABLE 4.19

Analysis of Variance for a Graeco-Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom
Latin letter treatments	$SS_L = \frac{1}{p}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$
Greek letter treatments	$SS_G = \frac{1}{p}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p-1$
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{l=1}^{p} y_{...l}^2 - \frac{y_{....}^2}{N}$	$p-1$
Error	SS_E (by subtraction)	$(p-3)(p-1)$
Total	$SS_T = \sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	p^2-1

Rocket Propellant Experiment with Graeco-Latin Square Design

- In the rocket propellant experiment from Example 4.3, an additional factor, *test assemblies*, could be of importance.
- Let there be five test assemblies, denoted by the Greek letters: α, β, γ, δ, and ϵ.
- The resulting 5×5 Graeco-Latin square design is displayed in Table 4.20.
- Note: The totals for batches of raw material (rows), operators (columns), and formulations (Latin letters) remain identical to those in Example 4.3.

Rocket Propellant Experiment with Graeco-Latin Square Design

TABLE 4.20
Graeco-Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators					
	1	2	3	4	5	$y_{i..}$
1	$A\alpha = -1$	$B\gamma = -5$	$C\epsilon = -6$	$D\beta = -1$	$E\delta = -1$	-14
2	$B\beta = -8$	$C\delta = -1$	$D\alpha = 5$	$E\gamma = 2$	$A\epsilon = 11$	9
3	$C\gamma = -7$	$D\epsilon = 13$	$E\beta = 1$	$A\delta = 2$	$B\alpha = -4$	5
4	$D\delta = 1$	$E\alpha = 6$	$A\gamma = 1$	$B\epsilon = -2$	$C\beta = -3$	3
5	$E\epsilon = -3$	$A\beta = 5$	$B\delta = -5$	$C\alpha = 4$	$D\gamma = 6$	7
$y_{..l}$	-18	18	-4	5	9	$10 = y_{...}$

Rocket Propellant Experiment with Graeco-Latin Square Design

The totals for the test assemblies (Greek letters) are

Greek Letter	Test Assembly Total
α	$y_{..1.} = 10$
β	$y_{..2.} = -6$
γ	$y_{..3.} = -3$
δ	$y_{..4.} = -4$
ϵ	$y_{..5.} = 13$

Rocket Propellant Experiment with Graeco-Latin Square Design

Thus, the sum of squares due to the test assemblies is

$$SS_{\text{Assemblies}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$$
$$= \frac{1}{5}[10^2 + (-6)^2 + (-3)^2 + (-4)^2 + 13^2] - \frac{(10)^2}{25} = 62.00$$

Rocket Propellant Experiment with Graeco-Latin Square Design

- The complete ANOVA is summarized in Table 4.21.
- Formulations are significantly different at 1 percent.
- In comparing Tables 4.21 and 4.12, we observe that removing the variability due to test assemblies has decreased the experimental error. However, in decreasing the experimental error, we have also reduced the error degrees of freedom from 12 (in the Latin square design of Example 4.3) to 8.
- Thus, our estimate of error has fewer degrees of freedom, and the test may be less sensitive.

Rocket Propellant Experiment with Graeco-Latin Square Design

TABLE 4.21
Analysis of Variance for the Rocket Propellant Problem

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Formulations	330.00	4	82.50	10.00	0.0033
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Test assemblies	62.00	4	15.50		
Error	66.00	8	8.25		
Total	676.00	24			

Balanced Incomplete Block Designs

Balanced Incomplete Block Designs (BIBDs)

Incomplete Block Designs

In certain experiments, not all treatment combinations can be run in each block due to limitations in experimental apparatus or facilities.

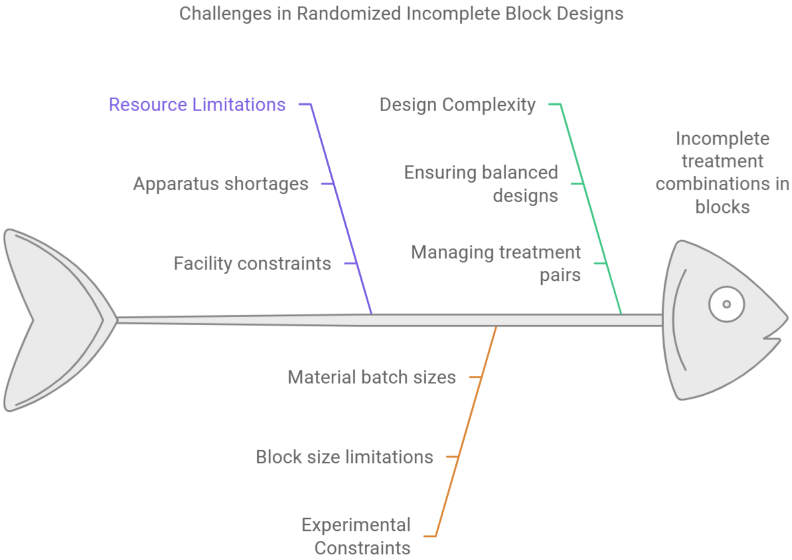
Balanced Allocation

BIBDs ensure that all treatment comparisons are equally important by balancing the allocation of treatments within blocks, so that any pair of treatments occurs together the same number of times.

Parameters

BIBDs are characterized by parameters such as the number of treatments (a), the number of blocks (b), the number of treatments per block (k), the number of times each treatment appears (r), and the number of times each pair of treatments appears together (λ).

Balanced Incomplete Block Designs



Statistical Analysis of the BIBD

Statistical Analysis of BIBDs

1 Model

The statistical model for a BIBD assumes that the observations are affected by treatment effects, block effects, and random error.

2 Partitioning Variability

The total variability in the data is partitioned into sums of squares for treatments (adjusted), blocks, and error.

3 F-test

The F-test is used to test the equality of treatment effects, comparing the mean square for treatments (adjusted) to the mean square for error.

Statistical Analysis of the BIBD

As usual, we assume that there are a treatments and b blocks. In addition, we assume that each block contains k treatments, that each treatment occurs r times in the design (or is replicated

Key Parameters in BIBD

- Each treatment appears r times.
- Total number of observations: $N = \bar{a}\bar{b}k$.
- Number of times each pair of treatments appears in the same block: λ.
- If $a = b$, the design is symmetric.

Deriving the Relationship for λ

- For a treatment (e.g., Treatment 1) appearing in r blocks:

$$r(k-1) = \lambda(a-1)$$

- Rearranging gives:

$$\lambda = \frac{r(k-1)}{a-1}$$

- The parameter λ must be an integer.

Statistical Model for BIBD

- The model is expressed as:

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$$

where:

  - y_{ij}: i-th observation in the j-th block.
  - μ: Overall mean.
  - τ_i: Effect of the i-th treatment.
  - β_j: Effect of the j-th block.
  - ϵ_{ij}: NID$(0, \sigma^2)$ random error.

Partitioning Total Variability

- Total corrected sum of squares:

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{N}$$

- Partitioning:

$$SS_T = SS_{\text{Treatments(adjusted)}} + SS_{\text{Blocks}} + SS_E$$

- Adjusted treatment sum of squares:

$$SS_{\text{Treatments(adjusted)}} = \frac{1}{k} \sum_{i=1}^{a} Q_i^2$$

where:

$$Q_i = y_{i.} - \frac{1}{k} \sum_{j=1}^{b} n_{ij} y_{.j}$$

Degrees of Freedom and F-Test

- Degrees of freedom:
  - SS_{Blocks}: $b-1$.
  - $SS_{\text{Treatments(adjusted)}}$: $a-1$.
  - SS_E: $N-a-b+1$.
- Error sum of squares:

$$SS_E = SS_T - SS_{\text{Treatments(adjusted)}} - SS_{\text{Blocks}}$$

- F-statistic for testing equality of treatment effects:

$$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$$

Statistical Analysis of the BIBD

TABLE 4.23
Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\frac{k\sum Q_i^2}{\lambda a}$	$a-1$	$\frac{SS_{\text{Treatments(adjusted)}}}{a-1}$	$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\frac{1}{k}\sum y_{.j}^2 - \frac{y_{..}^2}{N}$	$b-1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$N-a-b+1$	$\frac{SS_E}{N-a-b+1}$	
Total	$\sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$	$N-1$		

Example of the BIBD

TABLE 4.22

Balanced Incomplete Block Design for Catalyst Experiment

Treatment (Catalyst)	Block (Batch of Raw Material)				
	1	**2**	**3**	**4**	$y_{i.}$
1	73	74	—	71	218
2	—	75	67	72	214
3	73	75	68	—	216
4	75	—	72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

Example of the BIBD

EXAMPLE 4.5

Consider the data in Table 4.22 for the catalyst experiment. This is a BIBD with $a = 4$, $b = 4$, $k = 3$, $r = 3$, $\lambda = 2$, and $N = 12$. The analysis of this data is as follows. The total sum of squares is

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{12}$$
$$= 63{,}156 - \frac{(870)^2}{12} = 81.00$$

The block sum of squares is found from Equation 4.33 as

$$SS_{\text{Blocks}} = \frac{1}{3}\sum_{j=1}^{4} y_{.j}^2 - \frac{y_{..}^2}{12}$$
$$= \frac{1}{3}[(221)^2 + (207)^2 + (224)^2 + (218)^2] - \frac{(870)^2}{12}$$
$$= 55.00$$

To compute the treatment sum of squares adjusted for blocks, we first determine the adjusted treatment totals using Equation 4.35 as

$$Q_1 = (218) - \tfrac{1}{3}(221 + 224 + 218) = -9/3$$
$$Q_2 = (214) - \tfrac{1}{3}(207 + 224 + 218) = -7/3$$
$$Q_3 = (216) - \tfrac{1}{3}(221 + 207 + 224) = -4/3$$
$$Q_4 = (222) - \tfrac{1}{3}(221 + 207 + 218) = 20/3$$

The adjusted sum of squares for treatments is computed from Equation 4.34 as

$$SS_{\text{Treatments(adjusted)}} = \frac{k\sum_{i=1}^{4} Q_i^2}{\lambda a}$$
$$= \frac{3[(-9/3)^2 + (-7/3)^2 + (-4/3)^2 + (20/3)^2]}{(2)(4)}$$
$$= 22.75$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments(adjusted)}} - SS_{\text{Blocks}}$$
$$= 81.00 - 22.75 - 55.00 = 3.25$$

The analysis of variance is shown in Table 4.24. Because the P-value is small, we conclude that the catalyst employed has a significant effect on the time of reaction.

TABLE 4.24

Example of the BIBD

TABLE 4.24
Analysis of Variance for Example 4.5

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Treatments (adjusted for blocks)	22.75	3	7.58	11.66	0.0107
Blocks	55.00	3	—		
Error	3.25	5	0.65		
Total	81.00	11			